Modified Scaling Relation for the Random-Field Ising Model

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We investigate the low-temperature critical behavior of the three dimensional random-field Ising ferromagnet. By a scaling analysis we find that in the limit of temperature $T \to 0$ the usual scaling relations have to be modified as far as the exponent α of the specific heat is concerned. At zero temperature, the Rushbrooke equation is modified to $\alpha + 2\beta + \gamma = 1$, an equation which we expect to be valid also for other systems with similar critical behavior. We test the scaling theory numerically for the three dimensional random field Ising system with Gaussian probability distribution of the random fields by a combination of calculations of exact ground states with an integer optimization algorithm and Monte Carlo methods. By a finite size scaling analysis we calculate the critical exponents $\nu \approx 1.0$, $\beta \approx 0.05$, $\bar{\gamma} \approx 2.9$ $\gamma \approx 1.5$ and $\alpha \approx -0.55$.

Above two dimensions, the ferromagnetic random-field Ising model has an ordered phase for low temperatures and small random-fields as was proven by Imbrie [1] and later also by Bricmont and Kupiainen [2]. For larger fields the system develops a domain state [3] which has been shown to have a complex and fractal structure [4]. It is now widely believed that the phase transition from the ordered to the disordered phase is of second order. In three dimensions, the values of some of the critical exponents are now well established, like $\beta \approx 0.05$ and $\nu \approx 1$. Although real space renormalization yields deviating results concerning ν (see e. g. [5]) the values for β/ν are in the same range. However, a complete set of values of the critical exponents fulfilling the predicted set of scaling relations [6–8] could not be established, neither by experimental measurements (for a review, see [9]) performed usually on diluted antiferromagnets in a magnetic fields which are thought to be in the same universality class [10] nor by any numerical methods (see e. g. [11]). Especially, the value of α - and even its sign - is highly controversial.

The Hamiltonian of the RFIM in units of the nearest neighbor coupling constant J is

$$H = -\sum_{<ij>} \sigma_i \sigma_j - \sum_i B_i \sigma_i. \tag{1}$$

The first sum is over the nearest neighbors and the spin variables σ_i are ± 1. The random-fields B_i are taken from a Gaussian probability distribution $P(B_i) \sim \exp(-(B_i/\Delta)^2/2)$.

We assume that there is a zero temperature fixed point at a finite value Δ_c of the random-field width. Introducing the scaling variable $f = \Delta_0 - \Delta - g(T)$ (see also [7]) where the condition $f = 0$ describes the critical line we expect the same critical behavior no matter if we vary the temperature or the random-field strength. Hence, for the singular part of the internal energy E it should be

$$c \sim \frac{\partial E}{\partial T} \sim \frac{\partial E}{\partial \Delta} \sim |f|^{-\alpha}, \tag{2}$$

and for the singular part of the free energy F

$$c \sim -T \frac{\partial^2 F}{\partial T^2} \sim \frac{\partial^2 F}{\partial \Delta^2} \sim |f|^{-\alpha}. \tag{3}$$

In the limit of low temperatures F equals E and the question arises which derivative - first or second - with respect to the random-field strength yields the exponent of the specific heat?

The scaling ansatz for the singular part of the free energy for zero homogenous magnetic field is:

$$F(T, f) = |f|^{1/x_2} \mathcal{F}^{\pm} \left(\frac{T}{|f|^{x_1/x_2}} \right) \tag{4}$$

Hence, for the most singular part of the specific heat it is $c \sim |f|^{1/x_2 - 2}$ and consequently $\alpha = 2 - 1/x_2$, as usual. On the other hand, for fixed critical random-field $\Delta = \Delta_0$ and in the limit of temperature $T \to 0$ the prefactor T in Eq. 3 becomes critical and hence it is $\alpha = 1 - 1/x_2$. This is also consistent with the scaling behavior of $\frac{\partial E}{\partial \Delta} = \frac{\partial}{\partial \Delta}(F + TS)$ with $S = -\frac{\partial F}{\partial T}$ which follows from the scaling ansatz for the free energy. The most relevant singular terms are:

$$\frac{\partial E}{\partial \Delta} = -T|f|^{1/x_2 - 2} \frac{\partial |f|}{\partial \Delta} \frac{\partial |f|}{\partial T} \frac{1 - x_2}{x_2^2} \mathcal{F}^{\pm} + |f|^{1/x_2 - 1} \frac{\partial |f|}{\partial \Delta} \frac{1}{x_2} \mathcal{F}^{\pm} + \ldots \tag{5}$$

In the limit $T \to 0$ the first term vanishes and only the second term is observed in the specific heat leading to $\alpha = 1 - 1/x_2$ as above. Consequently, it follows by standard scaling theory that the scaling relations in the limit of temperature $T \to 0$ have to be modified with respect to α. For zero temperature, the equation corresponding to the Rushbrooke equation has the form

$$2\beta + \gamma = 1 - \alpha. \tag{6}$$

Additionally, it is remarkable that in Eq. 5 the second more singular term is small for either small T or small $\partial |f| / \partial T$. Therefore, one can expect to observe the

anomalous zero temperature critical behavior as long as the critical line is flat.

A similar, although more complicated consideration holds for the specific heat. Building the derivative $c = -T\frac{\partial^2 F}{\partial T^2}$, the most relevant singular terms are:

$$c = -T|f|^{1/x_2-2}\frac{1-x_2}{x_2^2}(\frac{\partial |f|}{\partial T})^2\mathcal{F}^{\pm}$$
$$-T|f|^{1/x_2-1}\frac{1}{x_2}\frac{\partial^2 |f|}{\partial T^2}\mathcal{F}^{\pm} + \dots \qquad (7)$$

The most relevant term has the square of the slope $\frac{\partial |f|}{\partial T}$ as a prefactor while the next relevant term has the curvature $\frac{\partial^2 |f|}{\partial T^2}$ as a prefactor. If the critical line starts horizontal at $T = 0$ but with a finite curvature the most relevant term will be suppressed and the unusual less critical behavior will be observed, yielding $\alpha = 1 - 1/x_2$ for low temperatures. Note that in both cases, for c as well as for $\frac{\partial E}{\partial \Delta}$ the $T \to 0$ critical behavior is an inflection point since $\frac{\partial |f|}{\partial \Delta}$ and $\frac{\partial^2 |f|}{\partial T^2}$ change the sign at the critical point. However, for finite temperatures close to the critical point a crossover to the "normal" critical behavior can be expected.

In order to test these arguments numerically we consider the three dimensional RFIM and calculate exact ground states (EGS) using an optimization algorithm well known in graph-theory. The Ising-system is mapped on an equivalent transport network, and the maximum flow is calculated using the Ford-Fulkerson algorithm [12–14]. We used a simple cubic lattice with periodical boundary conditions and linear lattice sizes varying from $L = 6$ to $L = 20$. From the spin configurations of the ground state, we can calculate the magnetization $M = [m]_{\rm av}$, where $m = \frac{1}{L^3}\sum_i \sigma_i$, the internal energy, $E = [h]_{\rm av}$ where $h = \frac{1}{L^3}H$ and the disconnected susceptibility $\chi_{\rm dis} = L^3\left[m^2\right]_{\rm av}$, where the square brackets denote an average taken over 30-1900 random-field configurations, depending on the system size. The advantage of the numerical technique above is that it supplies equilibrium information. But on the other hand it is restricted to zero temperature. Therefore, we combine the EGS calculation with Monte Carlo (MC) methods. Starting at zero temperature with an EGS-spin configuration for a certain set of random-fields we heat the system slowly using the standard heat bath algorithm. Hence, we get MC data at low temperatures, $T < T_c/2$, which are close to equilibrium. We checked that by heating the system with decreasing heating rates until no further change of the data was visible. Using this MC method we can additionally calculate the susceptibility $\chi = \frac{L^3}{T}\left[\langle m^2\rangle - \langle m\rangle^2\right]_{\rm av}$, and the specific heat $c = \frac{L^3}{T^2}\left[\langle h^2\rangle - \langle h\rangle^2\right]_{\rm av}$, where the angles denote a thermal average.

From the scaling relations above follow the finite size scaling relations

$$M = L^{-\beta/\nu}\tilde{M}\left((\Delta-\Delta_c)L^{1/\nu}\right) \qquad (8)$$

for the magnetization and

$$\chi_{\rm dis} = L^{\bar{\gamma}/\nu}\tilde{\chi}\left((\Delta-\Delta_c)L^{1/\nu}\right) \qquad (9)$$

for the disconnected susceptibility.

Figure 1 shows the scaling plot for the magnetization data from EGS calculations as described above yielding $\Delta_0 = \Delta_c(T=0) = 2.37 \pm 0.05$, $\nu = 1.0 \pm 0.1$, and $\beta = 0.05 \pm 0.05$. These are values which are not surprising and in agreement with most of the previous work, especially the previous EGS calculations of Ogielski [14]. The error-bars are estimated since there is no straight-forward way to extract error-bars from a finite-size scaling plot.

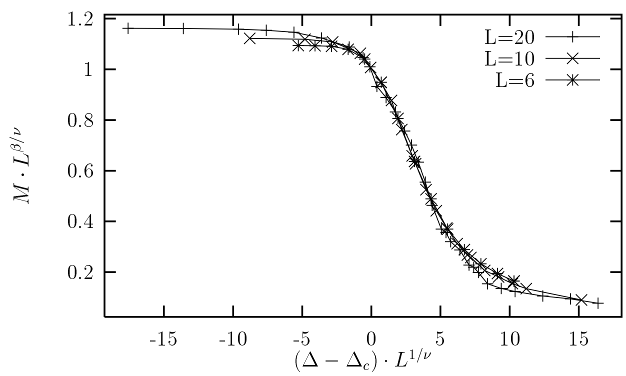

FIG. 1. Scaling plot of the magnetisation from EGS.

Using the same values for Δ_c and ν as above from the scaling plot for the disconnected susceptibility (not shown) we get $\bar{\gamma} = 2.9 \pm 0.3$ which is also in agreement with most of the previous work.

The first derivative of the ground state energy shows the critical behavior of the specific heat. The behavior of E can be understood by a series expansion of the energy in the vicinity of the critical point:

$$E(\Delta) = E_0 + E_1(\Delta-\Delta_c) + E_s(\Delta-\Delta_c)^{1-\alpha} + \dots \qquad (10)$$

The E_1-term is important since as we argued above α can be expected to be negative for low temperatures following Eq. 6. Hence, the finite size scaling form is

$$\frac{\partial E}{\partial \Delta} - E_1 = L^{\alpha/\nu}\tilde{E}\left((\Delta-\Delta_c)L^{1/\nu}\right) \qquad (11)$$

for the derivative of E in the critical region. Differentiating our energy data numerically, we obtained the scaling plot shown in Figure 2. Once more we used the same values for Δ_c and ν as above and chose E_1 such that $\frac{\partial E}{\partial \Delta} = E_1$ at the inflection point. Note, that we neglect here a possible size dependence of the analytic parts of the energy, i. e. a possible L-dependence of E_1 which is obviously very small as Figure 2 demonstrates. This analysis leads to $\alpha = -0.55 \pm 0.2$.

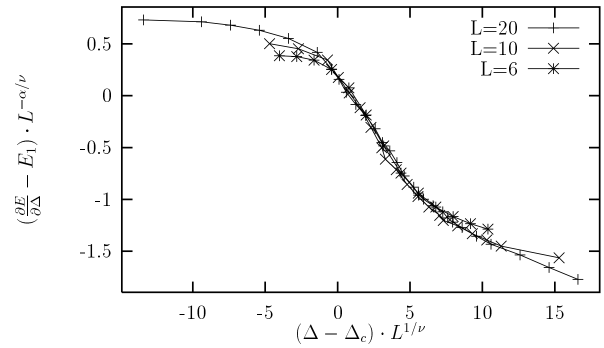

FIG. 2. Scaling plot of the derivative of energy from EGS.

The approach to determine α for finite temperatures is a direct MC simulation of the specific heat. Figure 3 shows the corresponding data. The ground state spin configurations were used as initial spin configurations for a MC simulation - of course of systems with identical random-field configuration. Then the systems were slowly heated (10000 MCS per temperature with temperature steps of 0.2). Data are shown for $T = 1.4$ which is roughly 30% of the critical temperature at zero field. We do not find any divergence of the specific heat, i. e. no size dependence of the maximum of c. Hence, as above we analyzed the data subtracting the value of the energy at the inflection point c_0. We took the values $\Delta_c(T = 1.4) = 2.35$ and $\nu = 1$ from MC simulation data of M for the same temperature (data not shown here). Our analysis yields once more $\alpha = -0.55 \pm 0.2$. The value $\Delta_c(T = 1.4)$ is very close to the zero-temperature value Δ_0, confirming that the critical line is nearly horizontal in the low temperature region. Hence, as discussed above the true critical behavior fulfilling the Rushbrooke equation is hard to observe and within our numerical accuracy we can only find the zero temperature exponent.

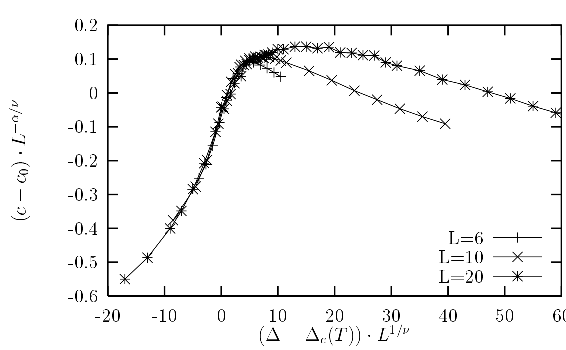

FIG. 3. Scaling plot of the specific heat from MC simulation. $T = 1.4$.

A standard finite size analysis of the MC data for χ yields $\gamma = 1.5 \pm 0.2$ (not shown). This value is in agreement with the previous work of Ogielski. Nevertheless there are deviations from previous results from series expansion where γ was found to be significantly higher [15]. We analyzed MC data also for other temperatures but we did not find any significant temperature dependence of the critical exponents in the temperature range $0 < T < 2$ which is nearly one half of the phase diagram.

To summarize, the values we determined for the critical exponents of the 3D RFIM are in good agreement with many of the previous works, experimental as well as theoretical. The exponents γ and $\bar{\gamma}$ fulfill the Schwartz-Soffer equation $\bar{\gamma} = 2\gamma$ [16]. The modified hyperscaling-relation [17] which can be written in the form (without α) $\bar{\gamma} = D\nu - 2\beta$ is also fulfilled by our exponents.

It is the aim of this work to calculate as many exponents as possible independently in order to test Eq.6 which is the most important aspect of our work. It is derived from the standard scaling ansatz for the zero temperature fixed point. Therefore we expect this equation to be valid also for other systems for which the discussed scaling ansatz is true. Candidates may be the random-field-Heisenberg model in appropriate dimensions. For the RFIM in higher dimensions new results suggest that there is a break of universality (i. e. the critical behavior depends on the kind of the distribution of the random-fields) as was shown in ref. [18] for four dimensions and earlier in refs. [19] and [20] for the mean field solution of the RFIM. Additionally, it was shown [21] that there is replica symmetry breaking for the mean field solution of a random field model with m-component-spins in the limit of large m. However, the replica symmetric solution of the RFIM with a Gaussian distribution of random fields has a zero temperature fixed point and indeed, in the limit $T \to 0$ the mean-field exponents fulfill eq. 6 since it is $\beta = 0.5$, $\gamma = 1$ and $\alpha = -1$ which can be inferred from a remark in ref. [19] stating that the entropy vanishes as $T \to 0$ linearly with T".

We argued that the crossover to the normal critical behavior might be hard to observe as long as the critical line is horizontal. We directly determined the controversially discussed exponent α for zero temperature yielding $\alpha = -0.55$ confirming the validity of Eq.6. The same value is also observed for finite but low temperatures, the crossover to the true critical behavior cannot be observed within our numerical accuracy. Surprisingly, this value is also in agreement with recent Monte Carlo simulations [11] as well as with recent experimental results [22]. Both did not find a divergence of the specific heat but $\alpha \leq 0$, although these measurements and simulation, respectively, were performed for higher temperatures where true critical behavior should be easier to observe.

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